CS103
Winter 2025

Lecture 16:

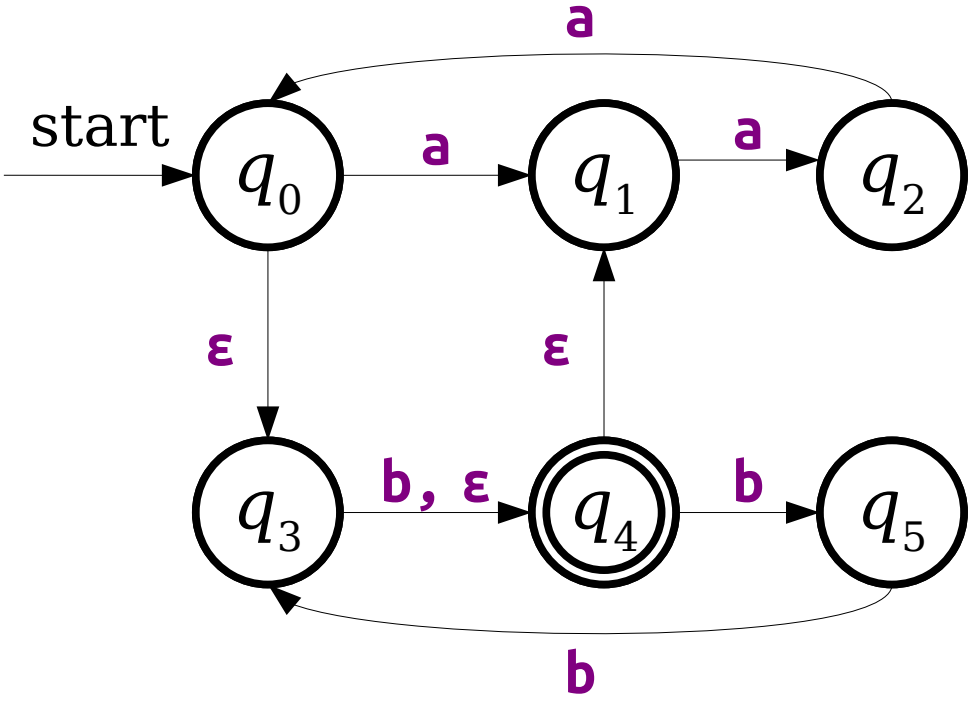
Finite Automata

Part 3 of 3

Recap from Last TIme

NFAs

- An ***NFA*** is a
  - ***N***ondeterministic
  - ***F***inite
  - ***A***utomaton
- NFAs have no restrictions on how many transitions are allowed per state.
- They can also use ε-transitions.
- An NFA accepts a string *w* if there is some sequence of choices that leads to an accepting state.



Massive Parallelism

- An NFA can be thought of as a DFA that can be in many states at once.
- At each point in time, when the NFA needs to follow a transition, it tries all the options at the same time.
- The NFA accepts if *any* of the states that are active at the end are accepting states. It rejects otherwise.

New Stuff!

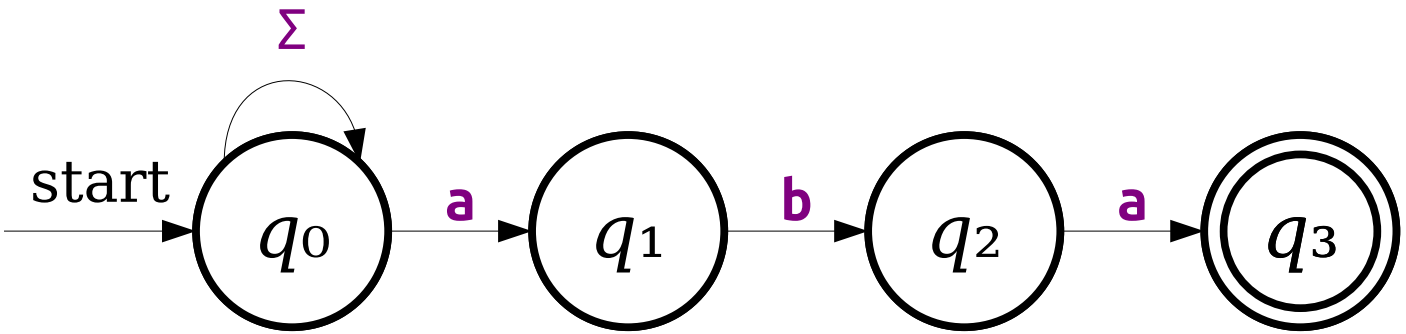
Just how powerful *are* NFAs?

NFAs and DFAs

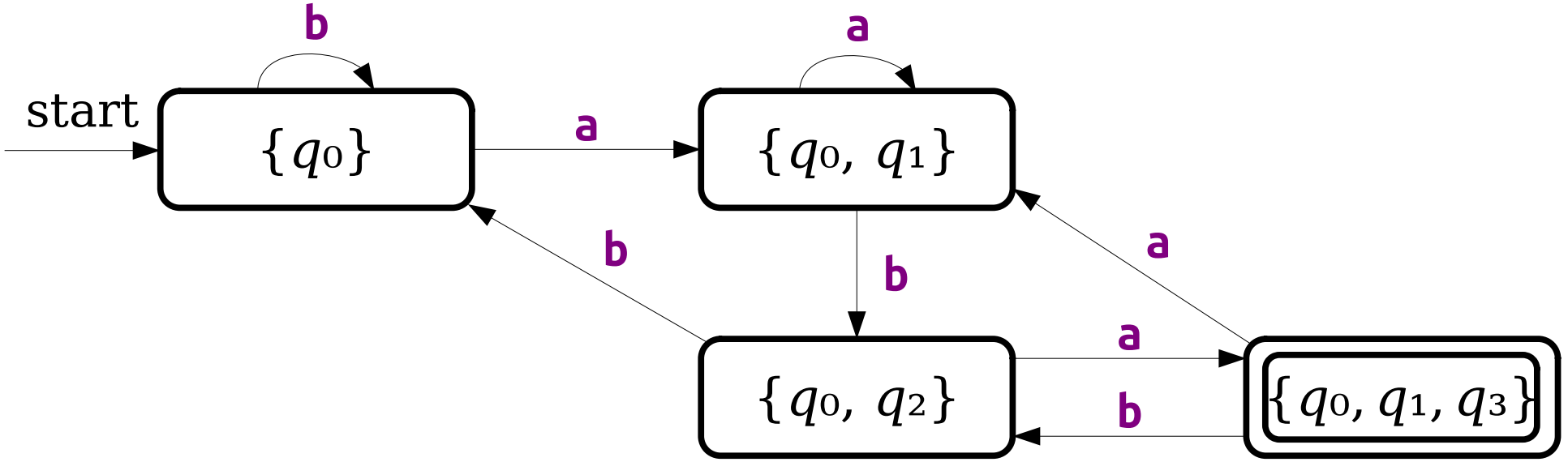
- Any language that can be accepted by a DFA can be accepted by an NFA.
- Why?
  - Every DFA essentially already *is* an NFA!
- ***Question:*** Can any language accepted by an NFA also be accepted by a DFA?
- Surprisingly, the answer is ***yes***!

Thought Experiment:

How would you simulate an NFA in software?



	a	b
$\{q_0\}$	$\{q_0, q_1\}$	$\{q_0\}$
$\{q_0, q_1\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$
$\{q_0, q_2\}$	$\{q_0, q_1, q_3\}$	$\{q_0\}$
$*\{q_0, q_1, q_3\}$	$\{q_0, q_1\}$	$\{q_0, q_2\}$



The Subset Construction

- This procedure for turning an NFA for a language L into a DFA for a language L is called the ***subset construction***.
  - It's sometimes called the ***powerset construction***; it's different names for the same thing!
- Intuitively:
  - Each state in the DFA corresponds to a set of states from the NFA.
  - Each transition in the DFA corresponds to what transitions would be taken in the NFA when using the massive parallel intuition.
  - The accepting states in the DFA correspond to which sets of states would be considered accepting in the NFA when using the massive parallel intuition.
- There's an online ***Guide to the Subset Construction*** with a more elaborate example involving ε-transitions and cases where the NFA dies; check that for more details.

The Subset Construction

- In converting an NFA to a DFA, the DFA's states correspond to sets of NFA states.
- ***Useful fact:*** $|\wp(S)| = 2^{|S|}$ for any finite set S.
- In the worst-case, the construction can result in a DFA that is *exponentially larger* than the original NFA.
- ***Question to ponder:*** Can you find a family of languages that have NFAs of size n, but no DFAs of size less than 2^n?

Regular Languages

- A language L is called ***regular*** when there's a DFA D that recognizes L (that is, $\mathscr{L}(D) = L$).
- ***Theorem:*** A language L is regular if and only if there's an NFA N that recognizes it (that is, $\mathscr{L}(N) = L$).
- This fact makes it possible to explore regular languages by considering either DFAs or NFAs.

Time-Out for Announcements!

Please see Sean’s post on Ed for today’s announcements.

Back to CS103!

Motivating Example: ***Numbers***

Numbers

- Numbers can be written in many ways:

2718

2,718

2.718×10^3

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etc.

- How would we design a DFA or NFA that checks if a particular string is a number in some numeral system?

Numbers

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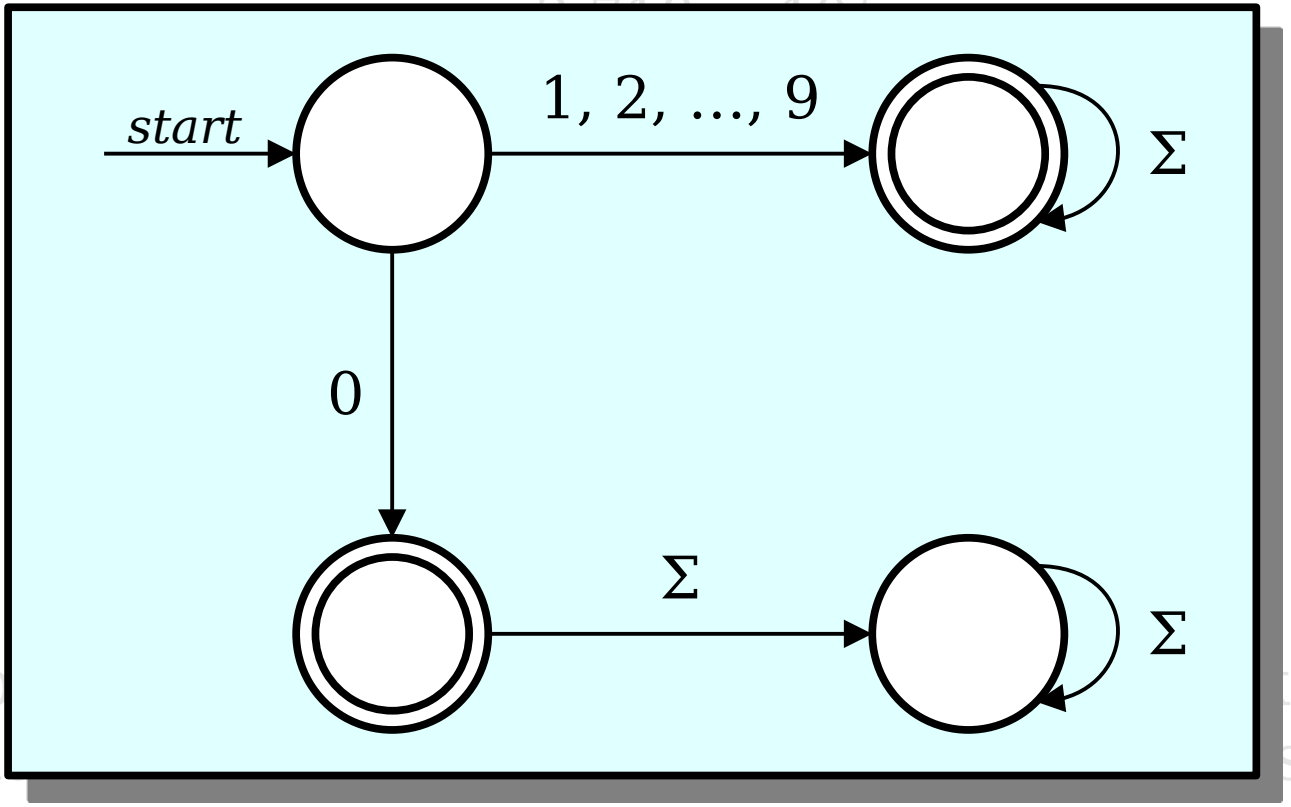
- How wo… …ts if a particul… …system?



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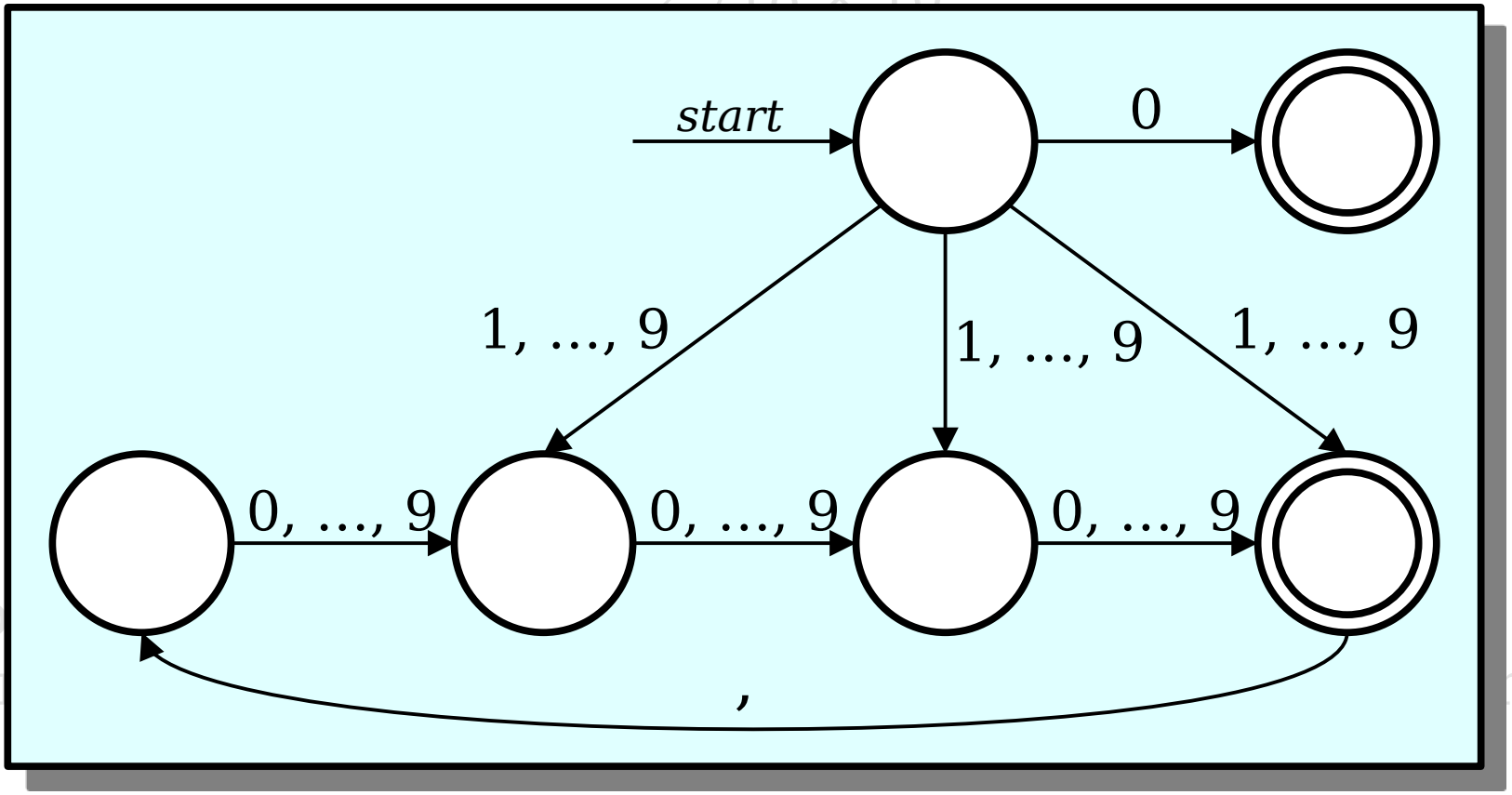
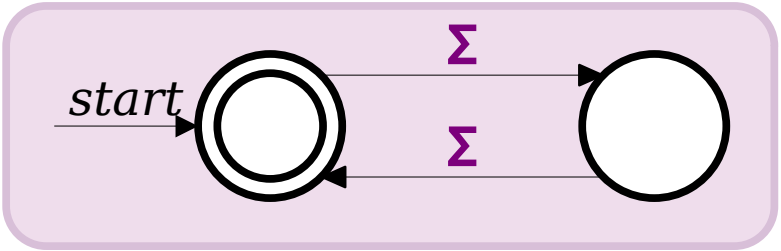


- How
  par m?

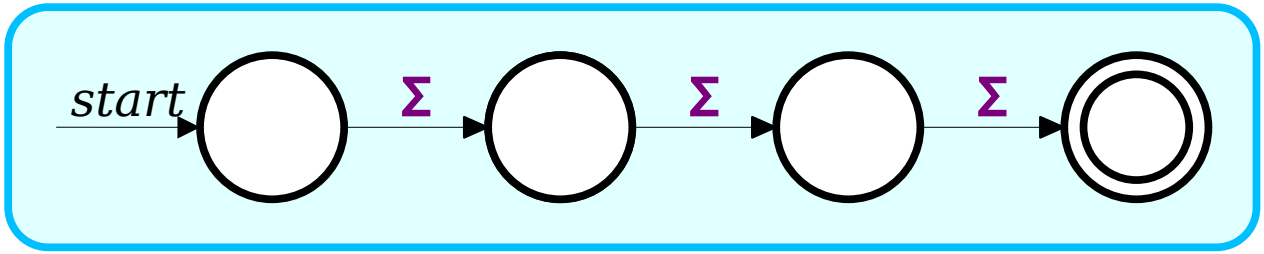
Practical Question: If we can build a bunch of finite automata that all recognize certain patterns, can we build a single finite automaton that recognizes all of those patterns?

Closure Under Union

- If L_1 and L_2 are languages over the alphabet Σ, the language $\boldsymbol{L_1 \cup L_2}$ is the language of all strings in at least one of the two languages.
- Intuitively, if L_1 and L_2 correspond to languages of strings with one of two different patterns, then $L_1 \cup L_2$ is the language of strings with at least one of those patterns.
- ***Theorem:*** If L_1 and L_2 are regular, so is $L_1 \cup L_2$.



DFA for L_1

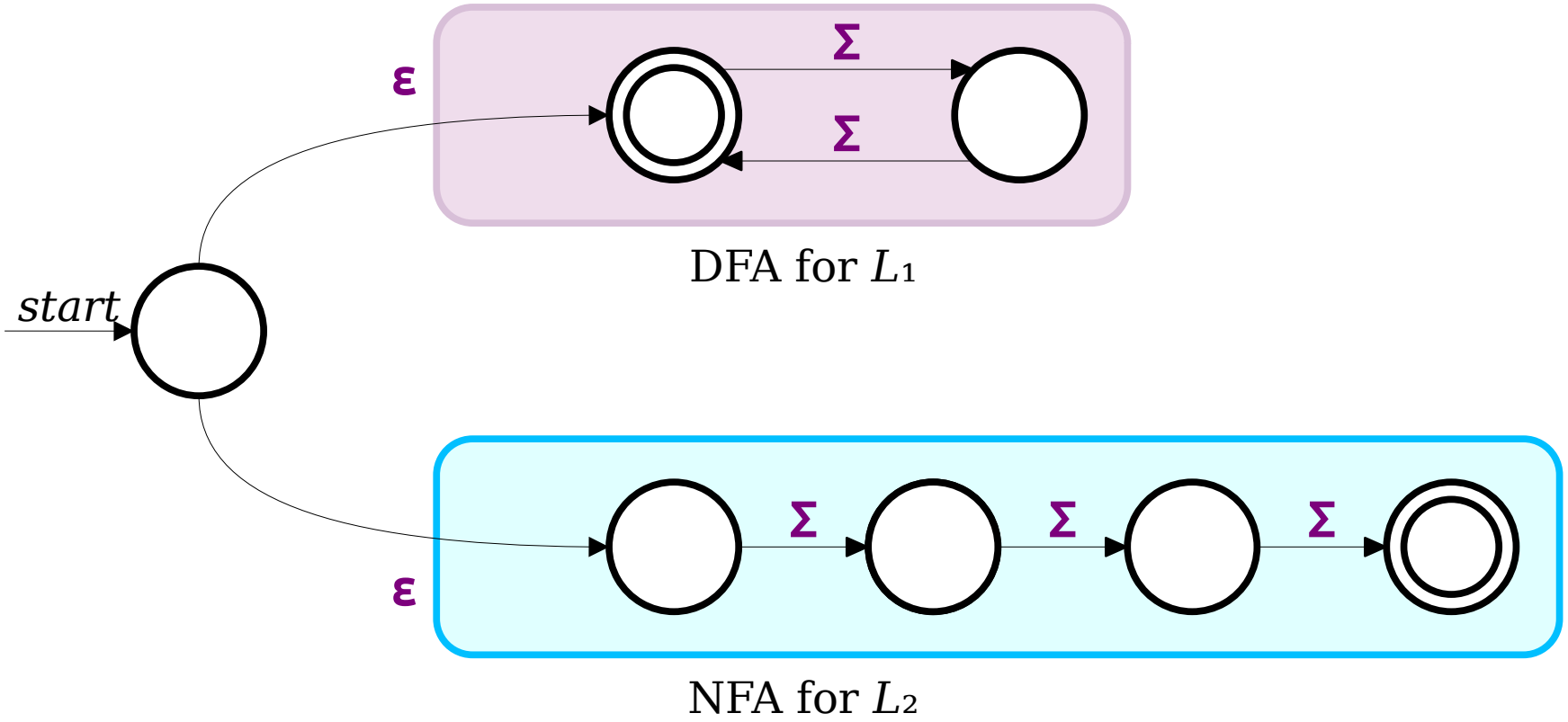


NFA for L_2

$L_1 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has even length} \}$

$L_2 = \{ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has length exactly three} \}$

Construct an NFA for $L_1 \cup L_2$.



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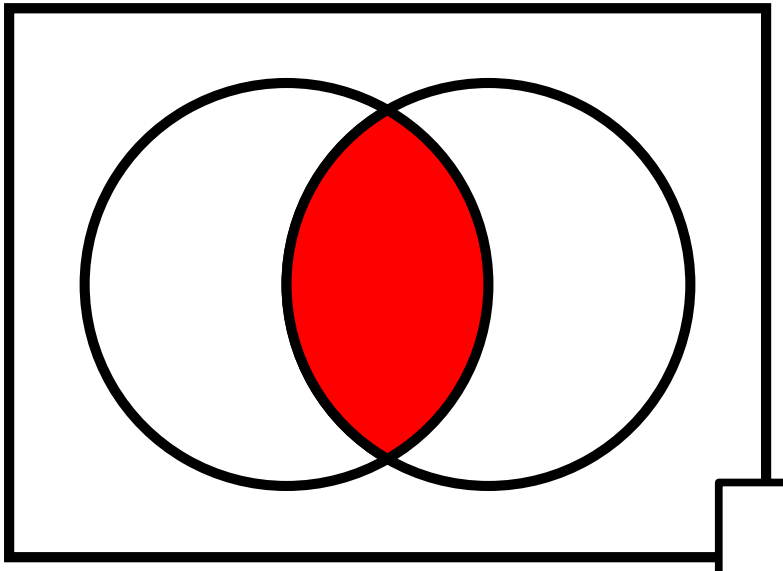
Construct an NFA for $L_1 \cup L_2$.

Closure Under Intersection

- If L_1 and L_2 are languages over Σ, then $\boldsymbol{L_1 \cap L_2}$ is the language of strings in both L_1 and L_2.
- Intuitively, $L_1 \cap L_2$ is the set of strings meeting the requirements of each language.
- ***Theorem:*** If L_1 and L_2 are regular, so is $L_1 \cap L_2$.



$$\overline{\overline{L_1} \cup \overline{L_2}}$$

Hey, it's De Morgan's laws!

Concatenation

Numbers

- Numbers can be written in many ways:

2718

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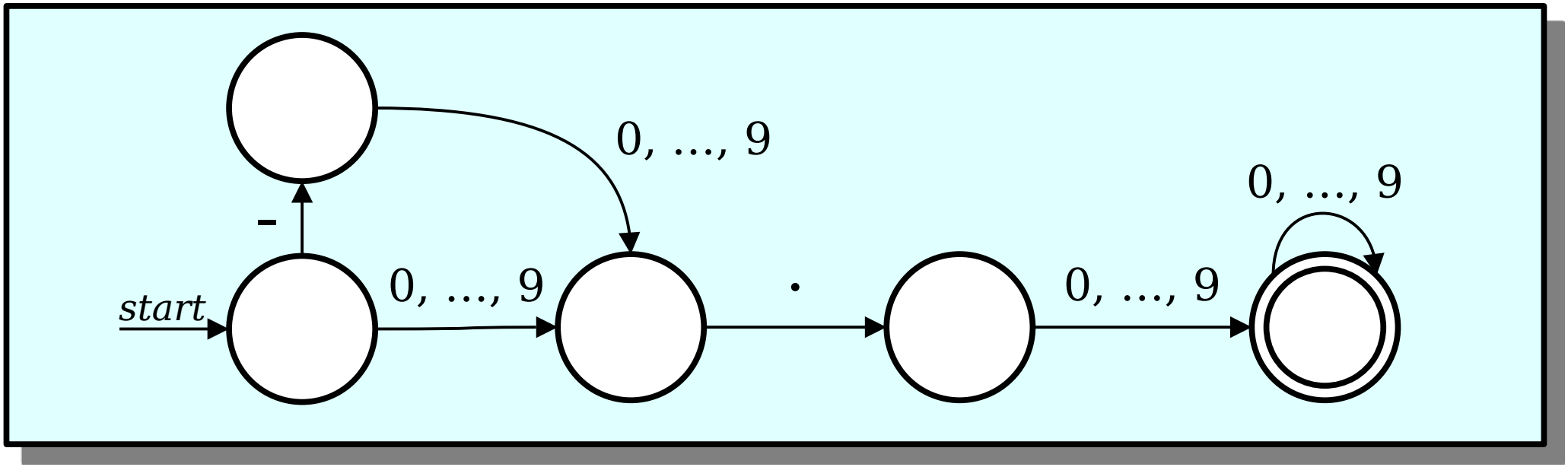
- How would we design a DFA or NFA that checks if a particular string is a number in some numeral system?



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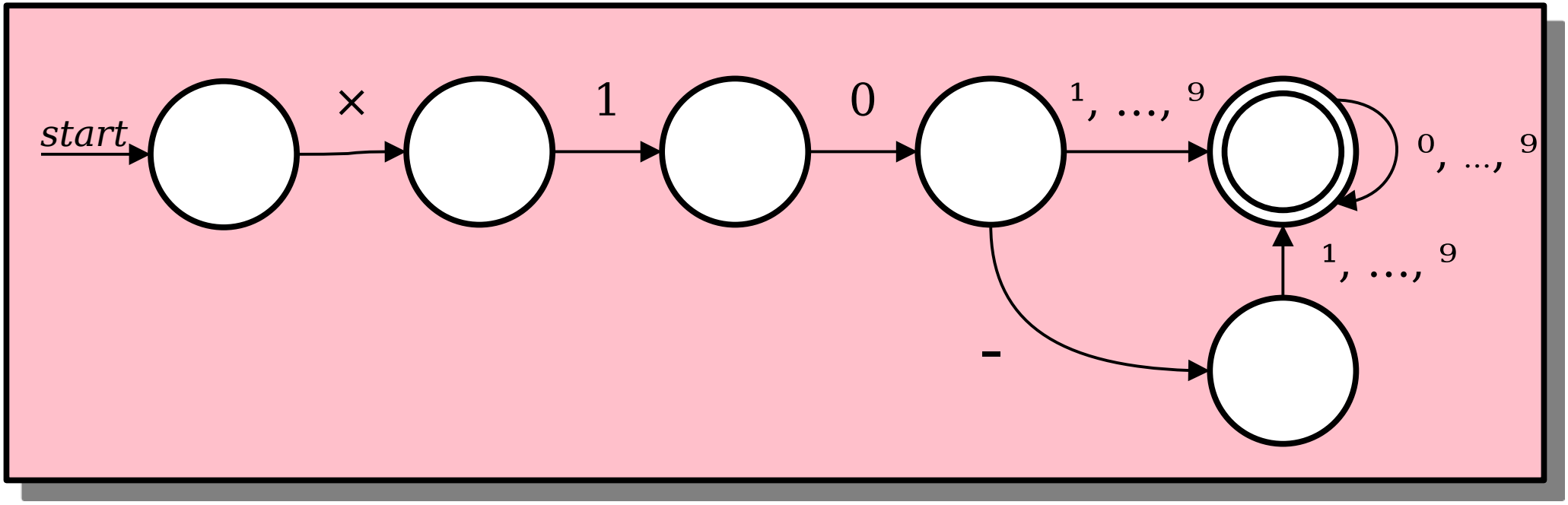


Question: If you can build finite automata to match the first and second halves of a pattern, can you build a single finite automaton that matches the full pattern?

String Concatenation

- If $w \in \Sigma^*$ and $x \in \Sigma^*$, the ***concatenation*** of w and x, denoted ***wx***, is the string formed by tacking all the characters of x onto the end of w.
- Example: if w = **quo** and x = **kka**, the concatenation wx = **quokka**.
- This is analogous to the + operator for strings in many programming languages.
- Some facts about concatenation:
  - The empty string ε is the ***identity element*** for concatenation:

    $$w\varepsilon = \varepsilon w = w$$

  - Concatenation is ***associative***:

    $$wxy = w(xy) = (wx)y$$

Concatenation

- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

  $$L_1L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2 \}$$

- Let $L_1 = \{$ **ab**, **ba** $\}$ and $L_2 = \{$ **aa**, **bb** $\}$. What is L_1L_2?

Answer at
https://cs103.stanford.edu/pollev

Concatenation Example

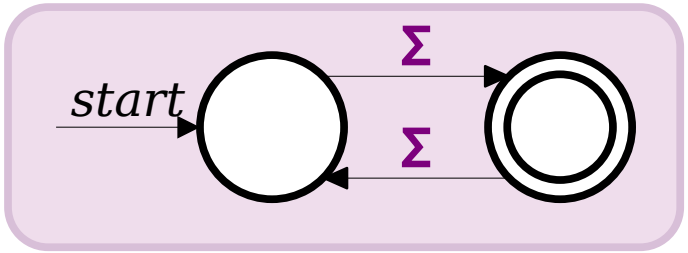
- Let Σ = { `a`, `b`, …, `z`, `A`, `B`, …, `Z` } and consider these languages over Σ:
  - ***Noun*** = { `Puppy`, `Rainbow`, `Whale`, … }
  - ***Verb*** = { `Hugs`, `Juggles`, `Loves`, … }
  - ***The*** = { `The` }
- The language ***TheNounVerbTheNoun*** is
  - { `ThePuppyHugsTheWhale`, `TheWhaleLovesTheRainbow`, `TheRainbowJugglesTheRainbow`, … }

Concatenation

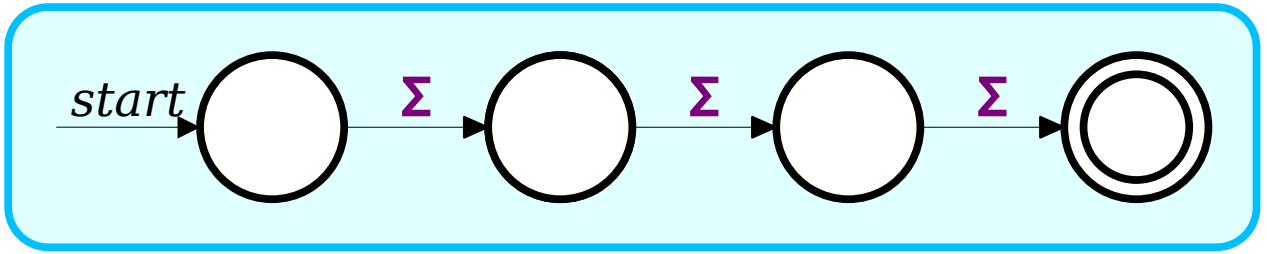
- The ***concatenation*** of two languages L_1 and L_2 over the alphabet Σ is the language

  $$L_1L_2 = \{ x \mid \exists w_1 \in L_1.\ \exists w_2 \in L_2.\ x = w_1w_2 \}$$

- Two views of L_1L_2:
  - The set of all strings that can be made by concatenating a string in L_1 with a string in L_2.
  - The set of strings that can be split into two pieces: a piece from L_1 and a piece from L_2.
- ***Theorem:*** If L_1 and L_2 are regular languages, then so is L_1L_2.



DFA for L_1

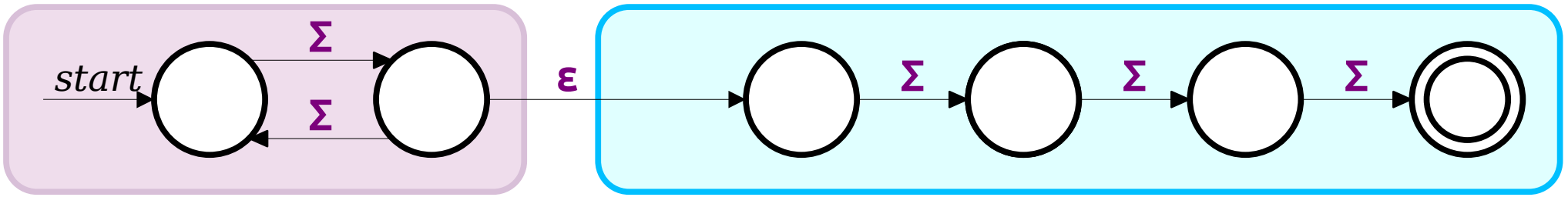


NFA for L_2

$L_1 = \{\ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has } \textit{odd} \text{ length}\ \}$

$L_2 = \{\ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w \text{ has length exactly three}\ \}$

Construct an NFA for L_1L_2.



DFA for L_1 NFA for L_2

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Construct an NFA for L_1L_2.

Numbers

- Suppose we successfully build a finite automaton that checks if a string is a numbers.
- Now, we want to make a new automaton that checks if a string consists of a *series* of numbers.
  - Perhaps we’re parsing a data file, for example.
- Do we have to start from scratch? Or could we reuse what we have?

The Kleene Star

Lots and Lots of Concatenation

- Consider the language L = { **aa**, **b** }
- LL is the set of strings formed by concatenating pairs of strings in L.

{ **aaaa**, **aab**, **baa**, **bb** }

- LLL is the set of strings formed by concatenating triples of strings in L.

{ **aaaaaa**, **aaaab**, **aabaa**, **aabb**, **baaaa**, **baab**, **bbaa**, **bbb**}

- $LLLL$ is the set of strings formed by concatenating quadruples of strings in L.

{ **aaaaaaaa**, **aaaaaab**, **aaaabaa**, **aaaabb**, **aabaaaa**,
aabaab, **aabbaa**, **aabbb**, **baaaaaa**, **baaaab**, **baabaa**,
baabb, **bbaaaa**, **bbaab**, **bbbaa**, **bbbb**}

Language Exponentiation

- We can define what it means to “exponentiate” a language as follows:
- $L^0 = \{\varepsilon\}$
  - Intuition: The only string you can form by gluing no strings together is the empty string.
  - Notice that $\{\varepsilon\} \neq \varnothing$. Can you explain why?
- $L^{n+1} = LL^n$
  - Idea: Concatenating (n+1) strings together works by concatenating n strings, then concatenating one more.
- ***Question to ponder:*** Why define $L^0 = \{\varepsilon\}$?
- ***Question to ponder:*** What is $\varnothing^0$?

The Kleene Closure

- An important operation on languages is the ***Kleene closure***, or ***Kleene star***, which is defined as

$$L^* = \{ w \in \Sigma^* \mid \exists n \in \mathbb{N}.\ w \in L^n \}$$

- Mathematically:

$$w \in L^* \quad \leftrightarrow \quad \exists n \in \mathbb{N}.\ w \in L^n$$

- Intuitively, L^* is the language all possible ways of concatenating zero or more strings in L together, possibly with repetition.
- ***Question to ponder:*** What is $\emptyset^*$?

The Kleene Closure

If L = { **a**, **bb** }, then L^* = {

ε,

a, **bb**,

aa, **abb**, **bba**, **bbbb**,

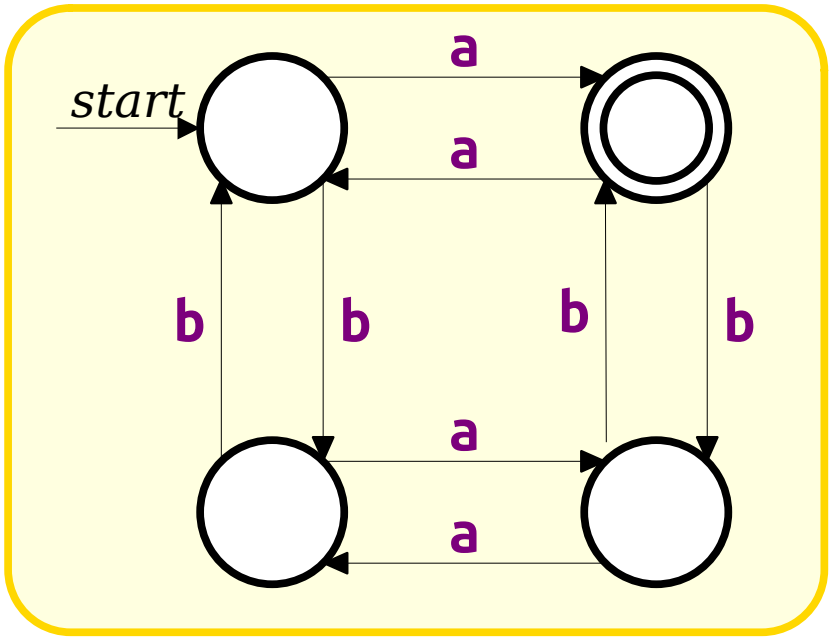
aaa, **aabb**, **abba**, **abbbb**, **bbaa**, **bbabb**, **bbbba**, **bbbbbb**,

...

}

Think of L* as the set of strings you can make if you have a collection of stamps – one for each string in L – and you form every possible string that can be made from those stamps.

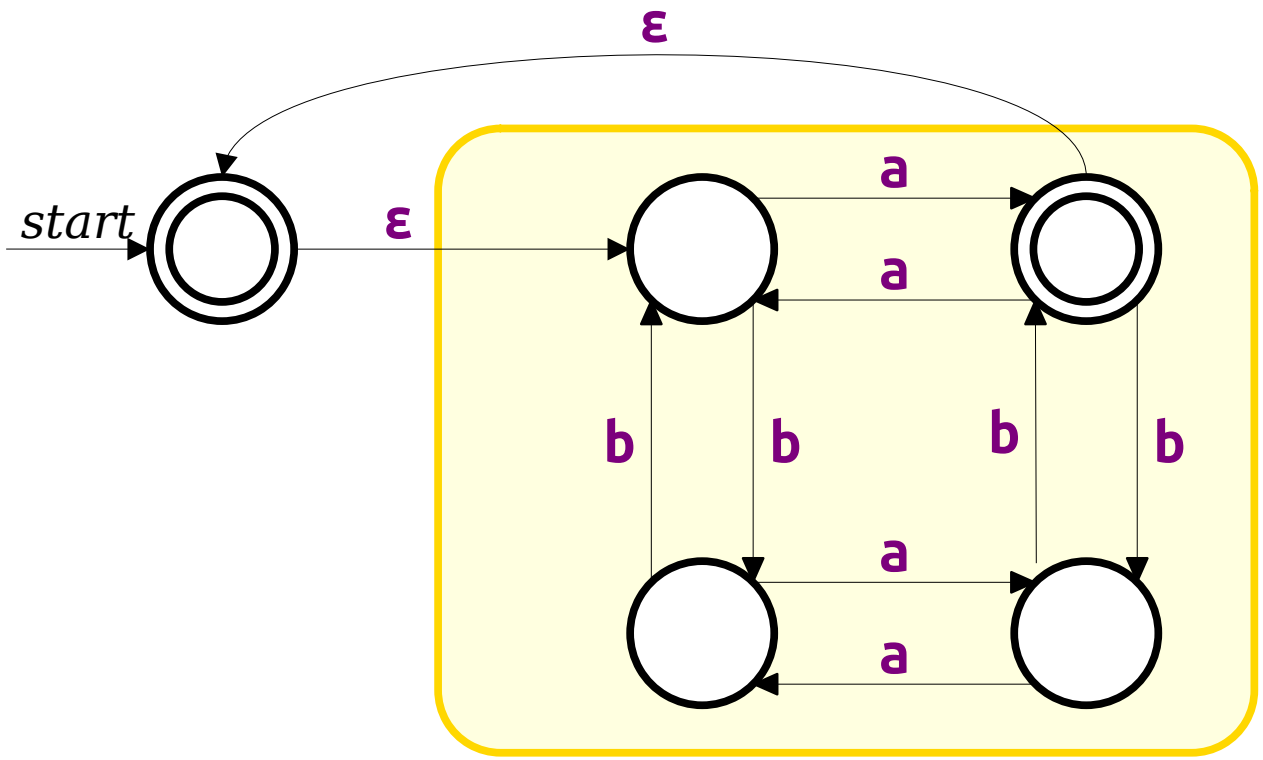
Theorem: If L is a regular language, so is L^*.



DFA for L

$L = \{\ w \in \{\mathtt{a}, \mathtt{b}\}^* \mid w$ has an odd number of **a**'s and an even number of **b**'s $\}$

Construct an NFA for L^*.



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Construct an NFA for L^*.

Closure Properties

- ***Theorem:*** If L_1 and L_2 are regular languages over an alphabet Σ, then so are the following languages:
  - $L_1 \cup L_2$
  - $L_1 \cap L_2$
  - $L_1 L_2$
  - L_1^*
- These are some of the ***closure properties of the regular languages***.

Next Time

- ***Regular Expressions***
  - Building languages from the ground up!
- ***Thompson's Algorithm***
  - A UNIX Programmer in Theoryland.
- ***Kleene's Theorem***
  - From machines to programs!